Find two linearly independent power series solutions of $(x+2)y'' - x^2 y' + 2xy = 0$ about the ordinary point $x = 0$.

SCORE: _____ / 12 PTS

You must find the recurrence relation for the coefficients and you must give the first three non-zero terms of each power series, but you do NOT need to write your final answers in sigma notation.

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(x+2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+1}$$

$$= \sum_{n=1}^{\infty} n(n+1)a_{n+1} x^n + \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} (n-1)a_{n-1} x^n + \sum_{n=1}^{\infty} 2a_{n-1} x^n$$

$$= \sum_{n=2}^{\infty} \left[n(n+1)a_{n+1} + 2(n+2)(n+1)a_{n+2} - (n-1)a_{n-1} + 2a_{n-1} \right] x^n$$

$$+ 2a_2 x + 4a_2 + 12a_3 x + 2a_0 x$$

EACH UNDERLINED ITEM WORTH 1 POINT UNLESS OTHERWISE LABELLED

$4a_2 = 0 \longrightarrow a_2 = 0$

$2a_2 + 12a_3 + 2a_0 = 0 \longrightarrow a_3 = -\frac{1}{6}a_0$

$n(n+1)a_{n+1} + 2(n+2)(n+1)a_{n+2} - (n-1)a_{n-1} + 2a_{n-1} = 0 \longrightarrow$

$$a_{n+2} = \frac{(n-3)a_{n-1} - n(n+1)a_{n+1}}{2(n+2)(n+1)}, \quad n \geq 2$$

$n=2: \quad a_4 = \dfrac{-a_1 - 6a_3}{24} = -\frac{1}{24}a_1 + \frac{1}{24}a_0$

$n=3: \quad a_5 = \dfrac{-12a_4}{40} = \frac{1}{80}a_1 - \frac{1}{80}a_0$

$y_1 = a_0\left(1 - \frac{1}{6}x^3 + \frac{1}{24}x^4 \ldots\right)$

$y_2 = a_1\left(x - \frac{1}{24}x^4 + \frac{1}{80}x^5 \ldots\right)$

$y = a_0 + a_1 x - \frac{1}{6}a_0 x^3 + \left(-\frac{1}{24}a_1 + \frac{1}{24}a_0\right)x^4 + \left(\frac{1}{80}a_1 - \frac{1}{80}a_0\right)x^5 + \ldots$

Use the method of Frobenius to find two linearly independent series solutions of $x^2(x+2)y'' - xy' + (1+x)y = 0$ about the regular singular point $x = 0$.

SCORE: _____ / 18 PTS

You must find the recurrence relation for the coefficients and you must give the first three non-zero terms of each series, but you do NOT need to write your final answers in sigma notation.

$$y'' - \frac{1}{x(x+2)}y' + \frac{1+x}{x^2(x+2)}y = 0$$

$$\lim_{x\to 0} x \cdot -\frac{1}{x(x+2)} = -\frac{1}{2} \qquad \lim_{x\to 0} x^2 \cdot \frac{1+x}{x^2(x+2)} = \frac{1}{2}$$

$$r(r-1) - \tfrac{1}{2}r + \tfrac{1}{2} = 0 \longrightarrow r(r-1) - \tfrac{1}{2}(r-1) = 0 \longrightarrow (r-\tfrac{1}{2})(r-1) = 0 \longrightarrow r = \tfrac{1}{2}, 1$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \qquad y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$(x^3+2x^2)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + (1+x)\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r+1} + \sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r+1}$$

$$= \sum_{n=1}^{\infty}(n+r-1)(n+r-2)a_{n-1}x^{n+r} + \sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r} + \sum_{n=1}^{\infty}a_{n-1}x^{n+r}$$

$$= \sum_{n=1}^{\infty}\left[(n+r-1)(n+r-2)a_{n-1} + 2(n+r)(n+r-1)a_n - (n+r)a_n + a_n + a_{n-1}\right]x^{n+r} + 2r(r-1)a_0x^r - ra_0x^r + a_0x^r$$

3

$2r(r-1) - r + 1 = 0$ SAME AS INDICIAL EQUATION ABOVE

$$(n+r-1)(n+r-2)a_{n-1} + 2(n+r)(n+r-1)a_n - (n+r)a_n + a_n + a_{n-1} = 0 \longrightarrow$$

$$a_n = -\frac{(n+r-1)(n+r-2)+1}{2(n+r)(n+r-1)-(n+r)+1}a_{n-1} = -\frac{(n+r-1)(n+r-2)+1}{(n+r-1)[2(n+r)-1]}a_{n-1}, \; n \geq 1$$

$r=1$: $a_n = -\dfrac{n(n-1)+1}{n(2n+1)}a_{n-1}$

$a_1 = -\frac{1}{3}a_0$

$a_2 = -\frac{3}{10}a_1 = \frac{1}{10}a_0$

$$y_1 = a_0x^1 - \tfrac{1}{3}a_0x^2 + \tfrac{1}{10}a_0x^3 = a_0(x - \tfrac{1}{3}x^2 + \tfrac{1}{10}x^3 \ldots)$$

$r=\frac{1}{2}$: $a_n = -\dfrac{(n-\frac{1}{2})(n-\frac{3}{2})+1}{(n-\frac{1}{2})[2(n+\frac{1}{2})-1]}a_{n-1} = -\dfrac{(n-\frac{1}{2})(n-\frac{3}{2})+1}{n(2n-1)}a_{n-1}$

$a_1 = -\frac{3}{4}a_0$

$a_2 = -\frac{7}{24}a_1 = \frac{7}{32}a_0$

$$y_2 = a_0x^{\frac{1}{2}} - \tfrac{3}{4}a_0x^{\frac{3}{2}} + \tfrac{7}{32}a_0x^{\frac{5}{2}} \ldots = a_0x^{\frac{1}{2}}(1 - \tfrac{3}{4}x + \tfrac{7}{32}x^2 \ldots)$$